

# Transition rate of the Unruh-DeWitt detector in curved spacetime

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## Abstract

We examine the Unruh-DeWitt particle detector coupled to a scalar field in an arbitrary Hadamard state in four-dimensional curved spacetime. Using smooth switching functions to turn on and off the interaction, we obtain a regulator-free integral formula for the total excitation probability, and we show that an instantaneous transition rate can be recovered in a suitable limit. Previous results in Minkowski space are recovered as a special case. As applications, we consider an inertial detector in the Rindler vacuum and a detector at rest in a static Newtonian gravitational field. Gravitational corrections to decay rates in atomic physics laboratory experiments on the surface of the Earth are estimated to be suppressed by 42 orders of magnitude.

## 1 Introduction

The Unruh-DeWitt model for a particle detector [1, 2] is an important tool for probing the physics of quantum fields wherever noninertial observers or curved backgrounds are present. In such cases there is often no distinguished notion of a “particle,” analogous to the plane-wave modes in Minkowski space, but an operational meaning can be attached to the concept by analysing the transitions induced among the energy levels of a detector coupled to the field. Upwards or downwards transitions can then be interpreted as due to absorption or emission of field quanta, or particles. The best-known applications

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of this procedure are those for which the spectrum of transitions is thermal, which is the case for uniformly accelerated detectors in Minkowski space [1], inertial detectors in de Sitter space [3], and detectors at rest in the exterior Schwarzschild black hole spacetime [4].

In first-order perturbation theory, the transition probability of the Unruh-DeWitt detector is proportional to a quantity known as the response function, which involves integrating the Wightman distribution of the quantum field over the worldline of the detector. When the quantum state of the field is sufficiently regular and the detector is switched on and off smoothly, the response function is well defined [5], and the physical interpretation is that the response function is then proportional to the probability of a transition to have occurred by a time at which all interaction has already ceased. If, however, one wishes to address the probability of a transition to have occurred by a time at which the interaction is still ongoing, the response function is no longer well defined because the switching function then has a sharp cut-off at a singularity of the Wightman distribution. In special cases in which the trajectory is stationary, the vacuum state is invariant under the Killing vector generating this stationary motion and the detector has been switched on in the infinite past [1, 2, 3, 4], the issue can be bypassed by formally integrating over the whole trajectory and factoring out the infinite total proper time, because by stationarity the transition rate can then be argued to be time-independent. But in a general setting this is not possible, and seemingly inconspicuous regularisations of the Wightman distribution can lead to unphysical results, even for uniformly accelerated motion in Minkowski space [6, 7].

A way to address this problem is to regard the sharp detector switch-off as a limit of a family of smooth switch-offs and investigate how the results depend on the way in which the limit is taken. In [8] this issue was investigated for a massless scalar field in four-dimensional Minkowski space, with the quantum field in the Minkowski vacuum. The response function with a smooth switching function was written in a form in which the integrand is no longer a distribution but a genuine function, and it was shown that a well-defined notion of a transition rate emerges when the switching time scale is small compared with the total duration of the coupling. It was also shown that in the appropriate limits this transition rate coincides with that obtained by regularising the sharply switched-off detector by a nonzero spatial size [6, 7]. The key point is that when the Wightman distribution under the integral is represented by an  $i\epsilon$ -regularised function, the regulator limit  $\epsilon \rightarrow 0$  and the limit to sharp switching do not in general commute and the first must be taken before the second.

The aim of this paper is to extend these results to a more general setting. For this, we will start in section 2 with a review of the Unruh-DeWitt detector, with special attention to the procedure introduced in [8] that allows limits of switching functions to be considered. In section 3 the results of [8] are generalised to a situation in which Minkowski space is replaced by an arbitrary four-dimensional globally hyperbolic spacetime, the Minkowski vacuum state by an arbitrary Hadamard state and the massless scalar field by a scalar field with arbitrary mass and curvature coupling. We shall in

particular obtain a simple and manifestly well-defined expression for the difference in the response of detectors that have the same switching function but move in different quantum states of the field, on different trajectories or even in different spacetimes. The limit of sharp switching is discussed in section 4. In sections 5 and 6 we use these results to obtain the detector transition rate in two examples of interest: an inertial detector in the Rindler vacuum in Minkowski space, and a detector at rest in a static Newtonian gravitational field. The results are summarised and discussed in section 7. Certain technical properties of the detector response in the Rindler vacuum are established in the Appendix.

Throughout this paper we will assume a Lorentzian metric of signature  $(-+++)$ , using the  $(+++)$  sign convention of Misner, Thorne and Wheeler [9]. We use units in which  $c = \hbar = 1$ , while keeping  $G = l_p^2 \neq 1$ . Spacetime points are denoted by sans-serif letters. The symbol  $O(x)$  denotes a quantity for which  $O(x)/x$  is bounded as  $x \rightarrow 0$ .  $O(1)$  denotes a quantity that is bounded in the limit under consideration.

## 2 Particle detectors and their regularisation

We consider a detector consisting of an idealised atom with two energy levels,  $|0\rangle_d$  and  $|1\rangle_d$ , with associated energy eigenvalues 0 and  $\omega$ . The detector is following a timelike  $C^\infty$  trajectory  $\mathbf{x}(\tau)$ , parametrised by its proper time  $\tau$ , in a four-dimensional Lorentzian globally hyperbolic  $C^\infty$  manifold  $M$ . The coupling of the detector to a real scalar field  $\phi$  of mass  $m$  and curvature coupling  $\xi$  is given by the interaction Hamiltonian

$$H_{\text{int}}(\tau) = c\chi(\tau)\mu(\tau)\phi(\mathbf{x}(\tau)), \quad (2.1)$$

where  $c$  is a coupling constant,  $\mu(\tau)$  is the detector's monopole moment operator and  $\chi(\tau)$  is a smooth non-negative function of compact support.  $\chi$  is called the *switching function*: the interaction takes place only when  $\chi$  is nonvanishing, and because  $\chi$  has compact support the interaction has a finite duration. If  $\mathbf{x}(\tau)$  is not defined for all  $\tau \in \mathbb{R}$ , we assume the support of  $\chi(\tau)$  to be in the open interval in which  $\mathbf{x}(\tau)$  is defined.

We take the initial state of the joint system before the interaction to be  $|\Psi\rangle \otimes |0\rangle_d$ , where the field state  $|\Psi\rangle$  is an arbitrary Hadamard state [10, 11]. We are interested in the probability for the detector to be observed at state  $|1\rangle_d$  after the interaction has been switched off. Treating the coupling constant  $c$  as a small parameter, working to first order in perturbation theory in  $c$ , and summing over the unobserved final state of the field, this probability reads [5, 12, 13]

$$P(\omega) = c^2 |{}_d\langle 0|\mu(0)|1\rangle_d|^2 F(\omega), \quad (2.2)$$

where the response function  $F(\omega)$  is given by

$$F(\omega) = \int_{-\infty}^{\infty} d\tau' \int_{-\infty}^{\infty} d\tau'' e^{-i\omega(\tau' - \tau'')} \chi(\tau') \chi(\tau'') W(\tau', \tau'') \quad (2.3)$$

and the distributional correlation function  $W(\tau', \tau'')$  is the pull-back of the Wightman distribution  $W(\mathbf{x}, \mathbf{x}') := \langle \Psi | \phi(\mathbf{x}) \phi(\mathbf{x}') | \Psi \rangle$  to the detector worldline,<sup>1</sup>

$$W(\tau', \tau'') := W(\mathbf{x}(\tau'), \mathbf{x}(\tau'')) . \quad (2.4)$$

The response function thus encodes the properties that depend on the state  $|\Psi\rangle$  and the detector trajectory, while the prefactor in (2.2) is a constant that only depends on the detector's internal properties. We shall from now on suppress the prefactor and refer to the response function simply as the probability.

To summarise, (2.3) gives an unambiguous answer to the question “What is the probability of the detector being observed in the state  $|1\rangle_d$  after the interaction has ceased?”

The meaning of the distributional correlation function under the integral in (2.3) is somewhat subtle. Recall that the Wightman distribution  $W(\mathbf{x}, \mathbf{x}')$  in a Hadamard state can be represented by a family of functions [10, 11]

$$W_\epsilon(\mathbf{x}, \mathbf{x}') = \frac{1}{(2\pi)^2} \left[ \frac{\Delta^{1/2}(\mathbf{x}, \mathbf{x}')}{\sigma_\epsilon(\mathbf{x}, \mathbf{x}')} + v(\mathbf{x}, \mathbf{x}') \ln(\sigma_\epsilon(\mathbf{x}, \mathbf{x}')) + H(\mathbf{x}, \mathbf{x}') \right] , \quad (2.5)$$

where  $\epsilon$  is a positive parameter,  $\sigma(\mathbf{x}, \mathbf{x}')$  is the squared geodesic distance between  $\mathbf{x}$  and  $\mathbf{x}'$ ,  $\sigma_\epsilon(\mathbf{x}, \mathbf{x}') := \sigma(\mathbf{x}, \mathbf{x}') + 2i\epsilon[T(\mathbf{x}) - T(\mathbf{x}')] + \epsilon^2$  and  $T$  is any globally-defined future-increasing  $C^\infty$  function. The logarithm denotes the branch that is real-valued on the positive real axis and has the cut on the negative real axis.  $\Delta(\mathbf{x}, \mathbf{x}')$  is the Van Vleck determinant, which is smooth for sufficiently near-by  $\mathbf{x}$  and  $\mathbf{x}'$ , the function  $v(\mathbf{x}, \mathbf{x}')$  is a polynomial in  $\sigma(\mathbf{x}, \mathbf{x}')$ , and the function  $H(\mathbf{x}, \mathbf{x}')$  can be chosen  $C^m$  for arbitrarily large  $n$  by taking the degree of the polynomial  $v(\mathbf{x}, \mathbf{x}')$  sufficiently high. The  $i\epsilon$ -prescription in (2.5) defines the singular part of  $W(\mathbf{x}, \mathbf{x}')$ : the action of the Wightman distribution is obtained by integrating  $W_\epsilon(\mathbf{x}, \mathbf{x}')$  against test functions and taking the limit  $\epsilon \rightarrow 0$ , and this limit can be shown to be independent of the choice of the global time function  $T$ . Now, the distributional correlation function  $W(\tau, \tau')$  (2.4) is the pull-back of  $W(\mathbf{x}, \mathbf{x}')$  to the detector's worldline, which is a  $C^\infty$  submanifold. It follows that the action of the distribution  $W(\tau, \tau')$  is obtained by pulling back the function  $W_\epsilon(\mathbf{x}, \mathbf{x}')$  to the function  $W_\epsilon(\tau, \tau')$ , integrating  $W_\epsilon(\tau, \tau')$  against test functions and taking the limit  $\epsilon \rightarrow 0$  [5, 14, 15]. Formula (2.3) must thus be understood as

$$F(\omega) = \lim_{\epsilon \rightarrow 0} \int_{-\infty}^{\infty} d\tau' \int_{-\infty}^{\infty} d\tau'' e^{-i\omega(\tau' - \tau'')} \chi(\tau') \chi(\tau'') W_\epsilon(\tau', \tau'') , \quad (2.6)$$

where the integrand is now an ordinary function and the singular part of  $W(\tau, \tau')$  has been encoded in the  $i\epsilon$  prescription. As  $\overline{W_\epsilon(\mathbf{x}, \mathbf{x}')} = W_\epsilon(\mathbf{x}', \mathbf{x})$ , where the overline denotes

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<sup>1</sup>We denote both the spacetime Wightman distribution  $W(\mathbf{x}, \mathbf{x}')$  and its pull-back  $W(\tau, \tau')$  to the detector worldline by the same letter, writing out the arguments explicitly in places where ambiguity could arise.

complex conjugation, we have  $\overline{W_\epsilon(\tau', \tau'')} = W_\epsilon(\tau'', \tau')$ , and it follows that (2.6) can be written in the equivalent form [6, 8]

$$F(\omega) = 2 \lim_{\epsilon \rightarrow 0} \operatorname{Re} \int_{-\infty}^{\infty} du \chi(u) \int_0^{\infty} ds \chi(u-s) e^{-i\omega s} W_\epsilon(u, u-s). \quad (2.7)$$

Although formulas (2.6) and (2.7) are suitable for computing the detector's response, these formulas do not display a clear separation between those properties of the response that depend on the trajectory and the quantum state and those properties that only depend on the choice of the switching function. Neither do these formulas exhibit how the response depends on the proper time along a given trajectory. Several authors [6, 16, 17, 18] have therefore addressed the question: “If the detector is turned on at proper time  $\tau_0$  and read at proper time  $\tau$ , while the interaction is still on, what is the probability that the transition has taken place?” If issues of regularisation could be ignored, this would amount to adopting in (2.3) the switching function

$$\chi(\tau') = \Theta(\tau' - \tau_0) \Theta(\tau - \tau'), \quad (2.8)$$

where  $\Theta$  is the Heaviside function. The transition probability then becomes a function of the reading time  $\tau$  and can be written as

$$F_\tau(\omega) = 2 \operatorname{Re} \int_{\tau_0}^{\tau} du \int_0^{u-\tau_0} ds e^{-i\omega s} W(u, u-s), \quad (2.9)$$

and we can define the instantaneous transition rate as its derivative with respect to  $\tau$ ,

$$\dot{F}_\tau(\omega) = 2 \operatorname{Re} \int_0^{\Delta\tau} ds e^{-i\omega s} W(\tau, \tau-s), \quad (2.10)$$

where  $\Delta\tau := \tau - \tau_0$ .  $\dot{F}_\tau(\omega)$  thus represents the number of transitions per unit time in an ensemble of identical detectors. It is this instantaneous transition rate that one expects to have the  $\tau$ -independent Planckian spectrum in the Unruh effect in Minkowski spacetime and in its generalisations to curved spacetimes, once  $\tau_0$  is taken to  $-\infty$  to avoid transient effects. A comprehensive recent review of the Unruh effect can be found in [19].

We note in passing that the transition rate (2.10) is not directly related to transition rates that could be measured with a single ensemble of detectors. Given an ensemble of identical detectors on a given trajectory,  $F_\tau(\omega)$  gives the fraction of detectors that have undergone a transition when observed at time  $\tau$ , but as an observation alters the dynamics of the system,  $F_\tau(\omega)$  no longer has this interpretation after a first observation has been made. To measure  $\dot{F}_\tau(\omega)$ , one therefore needs a set of identical ensembles, such that each ensemble is used to measure  $F_\tau(\omega)$  at just a single value of  $\tau$ . Note in particular that  $\dot{F}_\tau(\omega)$  may well be negative at some values of  $\tau$  [20, 21].  $\dot{F}_\tau(\omega)$  may thus be difficult to measure operationally, but it is nevertheless of interest as a nonstationary generalisation of the transition rate that naturally arises in stationary situations.

Returning to formulas (2.9) and (2.10), the difficulty with them as written is that the ‘switching function’ (2.8) is not smooth. We are no longer guaranteed that replacing  $W(\tau, \tau')$  by  $W_\epsilon(\tau, \tau')$  in (2.9) and (2.10) and taking the limit  $\epsilon \rightarrow 0$  would give a result that is independent of the choice of the global time function in (2.5). Case studies have shown that the result depends on the choice of the time function for Minkowski vacuum in Minkowski space [6, 7, 20, 21] and the Euclidean vacuum in de Sitter space [20, 21], and the methods of Appendix A of [7] can be adapted to show that the same holds for arbitrary Hadamard states in an arbitrary spacetime. Formula (2.10) does therefore not provide a well-defined notion of an instantaneous transition rate.

One way to address this problem was introduced in [6] and further developed in [7, 20, 21]. The idea is to replace the correlation function  $W(\tau, \tau')$  in (2.10) by a correlation function in which the field operator has been smeared over a spacelike hypersurface orthogonal to the trajectory. The weight function in the smearing is characterised by a positive length parameter  $\epsilon$ , which acts as a regulator and corresponds physically to the spatial size of the detector in its instantaneous rest frame. At the end the pointlike detector limit  $\epsilon \rightarrow 0$  is taken. This scheme does not rely on the choice of a time function to regularise the Wightman distribution, and in Minkowski space the introduction of the spatial hypersurfaces is straightforward [6] and there are partial results regarding independence of the choice of the weight function [7]. An implementation of the scheme in de Sitter space was given in [20, 21, 22]. However, the spacelike surfaces introduced in Minkowski space in [6] are not easily generalisable to spacetimes without a high degree of symmetry, and it would seem desirable to attach a meaning to the instantaneous transition rate within the framework of the conventional regularisation of the Wightman distribution (2.5).

A way that stays fully within the conventional regularisation of (2.5), (2.6) and (2.7) was introduced in [8] in the special case of Minkowski spacetime, massless scalar field and the Minkowski vacuum state. Adopting a Lorentz frame with global Minkowski coordinates  $(t, \mathbf{x})$  and choosing  $t$  as the global time function, formula (2.7) for the transition probability becomes

$$F(\omega) = \frac{1}{2\pi^2} \lim_{\epsilon \rightarrow 0} \operatorname{Re} \int_{-\infty}^{\infty} du \chi(u) \int_0^{\infty} ds \chi(u-s) e^{-i\omega s} \frac{1}{(\Delta \mathbf{x})^2 + 2i\epsilon \Delta t + \epsilon^2}, \quad (2.11)$$

where  $(\Delta \mathbf{x})^2$  is the squared geodesic distance between  $\mathbf{x}(u)$  and  $\mathbf{x}(u-s)$  and  $\Delta t := t(u) - t(u-s)$ . The limit  $\epsilon \rightarrow 0$  can be computed explicitly, with the result [8]

$$\begin{aligned} F(\omega) = & -\frac{\omega}{4\pi} \int_{-\infty}^{\infty} du [\chi(u)]^2 + \frac{1}{2\pi^2} \int_0^{\infty} \frac{ds}{s^2} \int_{-\infty}^{\infty} du \chi(u) [\chi(u) - \chi(u-s)] \\ & + \frac{1}{2\pi^2} \int_{-\infty}^{\infty} du \chi(u) \int_0^{\infty} ds \chi(u-s) \left( \frac{\cos(\omega s)}{(\Delta \mathbf{x})^2} + \frac{1}{s^2} \right). \end{aligned} \quad (2.12)$$

When the switching function  $\chi$  equals 1 over an interval of length  $\Delta\tau$ , and the switch-on and switch-off each take place within an interval of length  $\delta$  with a profile that scales

with  $\delta$  but whose shape is otherwise fixed, the leading behaviour of the transition rate (defined as the derivative of  $F(\omega)$  (2.12) with respect to  $\Delta\tau$ ) at  $\delta \rightarrow 0$  is

$$\dot{F}_\tau(\omega) = -\frac{\omega}{4\pi} + \frac{1}{2\pi^2} \int_0^{\Delta\tau} ds \left( \frac{\cos(\omega s)}{(\Delta\mathbf{x})^2} + \frac{1}{s^2} \right) + \frac{1}{2\pi^2 \Delta\tau} + O(\delta), \quad (2.13)$$

where now  $(\Delta\mathbf{x})^2$  is the squared geodesic distance between  $\mathbf{x}(\tau)$  and  $\mathbf{x}(\tau-s)$ . In the limit  $\delta \rightarrow 0$ , the transition rate (2.13) agrees with that obtained from spatial smearing in [7], and it reproduces the expected Planckian spectrum when the trajectory is uniformly linearly accelerated and  $\Delta\tau \rightarrow \infty$ . Further properties of this transition rate are discussed in [7, 8].

In this paper we generalise the Minkowski vacuum results (2.12) and (2.13) to a general Hadamard vacuum state in four-dimensional spacetime, for a field with arbitrary values of the mass and the curvature coupling. We shall show that most of the arguments in [8] carry over to this situation, and we shall find the expressions that generalise (2.12) and (2.13). These expressions will then be applied to two examples.

### 3 Regulator-free response function in a general Hadamard state

In this section we obtain a regulator-free expression for the response function  $F(\omega)$  by computing explicitly the limit  $\epsilon \rightarrow 0$  in (2.7). Following the procedure used in [8], we split the  $s$ -integral into the subintervals  $(0, \eta)$  and  $(\eta, \infty)$ , with  $\eta = \sqrt{\epsilon}$ , estimate the integrand in each subinterval and finally combine the results.

We shall make use of the small  $s$  expansions

$$\sigma = -s^2 - \frac{1}{12}a^2 s^4 + O(s^5), \quad (3.1a)$$

$$\Delta = 1 + O(s^2), \quad (3.1b)$$

$$v = m^2 + \left( \xi - \frac{1}{6} \right) R + O(s^2), \quad (3.1c)$$

$$\Delta T = \dot{T}s - \frac{\ddot{T}s^2}{2} + O(s^3), \quad (3.1d)$$

where the Ricci scalar  $R$ , the squared (covariant) acceleration  $a^2$  and  $T$  are evaluated at the point  $\mathbf{x}(u)$  and the dots indicate proper time derivatives.

#### 3.1 Subinterval $s \in (\eta, \infty)$

Consider in (2.7) the subinterval  $s \in (\eta, \infty)$ , and let  $W_0$  denote the pointwise limit of  $W_\epsilon$  as  $\epsilon \rightarrow 0$ . Replacing  $W_\epsilon$  by  $W_0$  creates under the  $u$ -integral an error that equals  $\chi(u)$

times the quantity

$$\begin{aligned} & 2 \operatorname{Re} \int_{\eta}^{\infty} ds \chi(u-s) e^{-i\omega s} [W_{\epsilon}(u, u-s) - W_0(u, u-s)] \\ &= \frac{1}{2\pi^2} \operatorname{Re} \int_{\eta}^{\infty} ds \chi(u-s) e^{-i\omega s} \left\{ \Delta^{1/2} \left( \frac{1}{\sigma_{\epsilon}} - \frac{1}{\sigma} \right) + v [\ln(\sigma_{\epsilon}) - \ln(\sigma)] \right\}, \end{aligned} \quad (3.2)$$

where the functions  $\Delta$ ,  $v$ ,  $\sigma$  and  $\sigma_{\epsilon}$  are each evaluated at the pair  $(\mathbf{x}, \mathbf{x}') = (\mathbf{x}(u), \mathbf{x}(u-s))$  and  $\ln(\sigma) := \lim_{\epsilon \rightarrow 0+} \ln(\sigma_{\epsilon})$ . We shall show that this error term does not contribute to (2.7) after the limit  $\epsilon \rightarrow 0$  is taken.

Consider first in (3.2) the contribution from  $\sigma_{\epsilon}^{-1}$  and  $\sigma^{-1}$ . We split this contribution into its odd and even parts in  $\omega$ . The part that is odd in  $\omega$  can be written as

$$-\frac{1}{2\pi^2} \int_{\eta}^{\infty} ds \chi(u-s) \sin(\omega s) \Delta^{1/2} \frac{2\epsilon \Delta T}{\sigma^2 \left[ \left( 1 + \frac{\epsilon^2}{\sigma} \right)^2 + \frac{4\epsilon^2 (\Delta T)^2}{\sigma^2} \right]}, \quad (3.3)$$

where  $\Delta T := T(\mathbf{x}(u)) - T(\mathbf{x}(u-s))$ . As the switching function makes the upper limit of the  $s$ -integral finite, it follows from (3.1a) and (3.1d) that the quantities  $(\Delta T)^2/\sigma$  and  $\epsilon/\sigma$  are bounded by constants that are independent of  $\eta$ . We can therefore write (3.3) as

$$-\frac{1}{\pi^2} \int_{\eta}^{\infty} ds \chi(u-s) \Delta^{1/2} \frac{\epsilon \sin(\omega s) \Delta T}{\sigma^2} [1 + O(\epsilon)], \quad (3.4)$$

where the  $O(\epsilon)$  estimate holds uniformly in  $s$ . As the functions  $\chi$  and  $\Delta$  are  $O(1)$  at small  $s$ , the integrand in (3.4) is bounded by a constant times  $\epsilon s^{-2}$ , and (3.4) is thus of order  $O(\epsilon \eta^{-1}) = O(\eta)$ . Similarly, the part that is even in  $\omega$  can be written as

$$-\frac{1}{2\pi^2} \int_{\eta}^{\infty} ds \chi(u-s) \cos(\omega s) \Delta^{1/2} \frac{\epsilon^2}{\sigma^2} \frac{1 + \frac{\epsilon^2}{\sigma} - \frac{4(\Delta T)^2}{\sigma}}{\left( 1 + \frac{\epsilon^2}{\sigma} \right)^2 + \frac{4\epsilon^2 (\Delta T)^2}{\sigma^2}}, \quad (3.5)$$

and similar estimates show that the integrand in (3.5) is bounded by a constant times  $\epsilon^2 s^{-4}$ . The expression (3.5) is hence of order  $O(\epsilon^2 \eta^{-3}) = O(\eta)$ .

Consider then in (3.2) the contribution from the logarithmic terms. Keeping track of the branches of the logarithms, we can write this contribution as

$$\frac{1}{2\pi^2} \operatorname{Re} \int_{\eta}^{\infty} ds \chi(u-s) e^{-i\omega s} v \ln \left( 1 + \frac{2i\epsilon \Delta T}{\sigma} + \frac{\epsilon^2}{\sigma} \right). \quad (3.6)$$

It follows from (3.1a) and (3.1d) that  $\epsilon \Delta T/\sigma$  is bounded by  $\eta$  times a constant and  $\epsilon^2/\sigma$  is bounded by  $\epsilon$  times a constant. The logarithm is hence of order  $O(\eta)$  uniformly in  $s$ , and (3.6) is of order  $O(\eta)$ .

As  $\chi$  has compact support, all the estimates above hold uniformly in  $u$  under the  $u$ -integral in (2.7). In the subinterval  $s \in (\eta, \infty)$  in (2.7),  $W_{\epsilon}$  can therefore be replaced by  $W_0$  without error.



### 3.2 Subinterval $s \in (0, \eta)$

We now turn to the subinterval  $s \in (0, \eta)$  in (2.7). The singularity of  $W$  at  $s = 0$  implies that we cannot directly replace  $W_\epsilon$  by  $W_0$ , and we will need to examine the small  $s$  behaviour of  $W_\epsilon$  more closely.

We observe first that the term  $H(\mathbf{x}, \mathbf{x}')$  in (2.5) clearly gives a vanishing contribution to (2.7).

Consider then the logarithmic term in (2.5). Suppressing for the moment the factor  $\chi(u)$ , the integral over  $u$  and the limit  $\epsilon \rightarrow 0$ , the contribution to (2.7) reads

$$\frac{1}{2\pi^2} \operatorname{Re} \int_0^\eta ds \chi(u-s) e^{-i\omega s} \left\{ \left[ m^2 + \left( \xi - \frac{1}{6} \right) R \right] + O(s^2) \right\} \ln(\sigma_\epsilon). \quad (3.7)$$

The imaginary part of the logarithm is bounded and its contribution in (3.7) is therefore of order  $O(\eta)$ . To estimate the real part of the logarithm, we write  $s = \epsilon x$ , with  $0 < x < 1/\eta$ , and use the expansions (3.1) to obtain

$$|\sigma_\epsilon|^2 = \epsilon^4 \left[ (1-x^2)^2 + 4x^2 \dot{T}^2 \right] [1 + O(\eta)], \quad (3.8)$$

where the  $O(\eta)$  term holds uniformly in  $x$ . Hence

$$2 \operatorname{Re} \ln(\sigma_\epsilon) = 4 \ln \epsilon + \ln \left[ (1-x^2)^2 + 4x^2 \dot{T}^2 \right] + O(\eta), \quad (3.9)$$

where again the  $O(\eta)$  term holds uniformly in  $x$ . The contribution in (3.7) is therefore of order  $O(\eta \ln \eta)$ . As this estimate holds uniformly in  $u$ , by virtue of the compact support of  $\chi$ , the logarithmic term does thus not contribute in (2.7).

Finally, consider the  $\sigma_\epsilon^{-1}$  term in (2.5). From (3.8) we see that  $s^2 \sigma_\epsilon^{-1}$  is bounded, and hence  $\int_0^\eta ds s^2 \sigma_\epsilon^{-1} = O(\eta)$ . It follows from (3.1b) that we may replace  $\Delta^{1/2}$  by 1, and we may similarly replace the factor  $\chi(u-s) e^{-i\omega s}$  by  $(1-i\omega s)\chi(u) - s\dot{\chi}(u)$ . What remains is to analyse the small  $\eta$  behaviour of the expression

$$I_< := \frac{1}{2\pi^2} \operatorname{Re} \int_0^\eta ds \frac{(1-i\omega s)\chi - s\dot{\chi}}{\sigma_\epsilon}, \quad (3.10)$$

where  $\chi$  and  $\dot{\chi}$  are evaluated at  $u$ . In the special case in which the spacetime is Minkowski space and the global time function  $T$  is the Minkowski time coordinate in a given Lorentz frame, this analysis was carried out in [8], and the techniques used therein generalise to (3.10) in a straightforward way. Splitting  $I_<$  into its even and odd parts in  $\omega$  as  $I_< = I_<^{\text{even}} + I_<^{\text{odd}}$ , and writing  $s = \epsilon x$  with  $0 < x < 1/\eta$ , we find<sup>2</sup>

$$I_<^{\text{even}} = \frac{1}{2\pi^2} \int_0^{1/\eta} \frac{(1-x^2) dx}{(1-x^2)^2 + 4x^2 \dot{T}^2} \left[ \frac{\chi}{\eta^2} - \dot{\chi}x + \frac{4\chi \dot{T} \ddot{T} x^3}{(1-x^2)^2 + 4x^2 \dot{T}^2} \right] + O(\eta), \quad (3.11a)$$

$$I_<^{\text{odd}} = -\frac{\omega \chi \dot{T}}{\pi^2} \int_0^{1/\eta} \frac{x^2 dx}{(1-x^2)^2 + 4x^2 \dot{T}^2} + O(\eta). \quad (3.11b)$$

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<sup>2</sup>Our formula (3.11b) corrects a typographical error in formula (3.4b) of [8].

In (3.11a) the integral of the first term is elementary, and multiplying the second and third term by  $\chi$  yields a total  $u$ -derivative that can be taken outside the integral. The result is

$$\chi I_{<}^{\text{even}} = \frac{\chi^2}{2\pi^2\eta} - \frac{1}{4\pi^2} \frac{d}{du} \int_0^{1/\eta} \frac{\chi^2 x(1-x^2) dx}{(1-x^2)^2 + 4x^2 \dot{T}^2} + O(\eta). \quad (3.12)$$

The integral in (3.11b) is elementary, and multiplying the result by  $\chi$  we obtain

$$\chi I_{<}^{\text{odd}} = -\frac{\omega \chi^2}{4\pi} + O(\eta). \quad (3.13)$$

All these estimates hold uniformly in  $u$ , owing to the compact support of  $\chi$ . The only terms that contribute in the subinterval  $s \in (0, \eta)$  in (2.7) are therefore the explicitly-displayed terms in (3.12) and (3.13).

### 3.3 Joining the subintervals

Substituting the results of subsections 3.1 and 3.2 in (2.7), we find

$$\begin{aligned} F(\omega) = & -\frac{\omega}{4\pi} \int_{-\infty}^{\infty} du [\chi(u)]^2 \\ & + 2 \lim_{\eta \rightarrow 0} \int_{-\infty}^{\infty} du \chi(u) \left[ \frac{\chi(u)}{4\pi^2\eta} + \text{Re} \int_{\eta}^{\infty} ds \chi(u-s) e^{-i\omega s} W_0(u, u-s) \right]. \end{aligned} \quad (3.14)$$

As  $\chi$  has compact support, the total derivative term in (3.12) integrates to zero and has dropped out. What remains is to take the limit in (3.14).

Following [8], we take the term proportional to  $1/\eta$  under the  $s$ -integral, add and subtract under the  $s$ -integral the term  $\chi(u-s)/(4\pi^2 s^2)$  and group the terms in the form

$$\begin{aligned} F(\omega) = & -\frac{\omega}{4\pi} \int_{-\infty}^{\infty} du [\chi(u)]^2 \\ & + 2 \lim_{\eta \rightarrow 0} \int_{-\infty}^{\infty} du \chi(u) \int_{\eta}^{\infty} ds \text{Re} \left[ \chi(u-s) e^{-i\omega s} W_0(u, u-s) + \frac{\chi(u)}{4\pi^2 s^2} \right] \\ = & -\frac{\omega}{4\pi} \int_{-\infty}^{\infty} du [\chi(u)]^2 \\ & + \lim_{\eta \rightarrow 0} \left\{ 2 \int_{-\infty}^{\infty} du \chi(u) \int_{\eta}^{\infty} ds \chi(u-s) \text{Re} \left[ e^{-i\omega s} W_0(u, u-s) + \frac{1}{4\pi^2 s^2} \right] \right. \\ & \quad \left. + \frac{1}{2\pi^2} \int_{\eta}^{\infty} \frac{ds}{s^2} \int_{-\infty}^{\infty} du \chi(u) [\chi(u) - \chi(u-s)] \right\}, \end{aligned} \quad (3.15)$$

where in the last term the interchange of the  $u$ -integral and the  $s$ -integral is justified by absolute convergence of the double integral. The limit  $\eta \rightarrow 0$  can now be taken

by simply setting  $\eta = 0$ . In the last term the reason is that the  $u$ -integral, when regarded as a function of  $s$ , has a Taylor expansion that starts with  $O(s^2)$ . In the term involving  $W_0$ , the reason is that the real part of  $e^{-i\omega s} W_0(u, u-s)$  has the small  $s$  behaviour of  $-1/(4\pi^2 s^2)$  plus an integrable function of  $s$ , by virtue of (2.5) and (3.1a). The final result for the response function is thus

$$F(\omega) = -\frac{\omega}{4\pi} \int_{-\infty}^{\infty} du [\chi(u)]^2 + \frac{1}{2\pi^2} \int_0^{\infty} \frac{ds}{s^2} \int_{-\infty}^{\infty} du \chi(u) [\chi(u) - \chi(u-s)] \\ + 2 \int_{-\infty}^{\infty} du \chi(u) \int_0^{\infty} ds \chi(u-s) \operatorname{Re} \left[ e^{-i\omega s} W_0(u, u-s) + \frac{1}{4\pi^2 s^2} \right]. \quad (3.16)$$

In the special case of the Minkowski vacuum in Minkowski space, (3.16) duly reduces to the expression (2.12) found in [8].

The first two terms in (3.16) depend only on the switching function  $\chi$  but neither on the quantum state, the spacetime or the trajectory. If we compare two detectors in different quantum states of the field, on different trajectories or even in different spacetimes, but having the same switching function, the difference of the responses is given by

$$\Delta F(\omega) = 2 \operatorname{Re} \int_{-\infty}^{\infty} du \chi(u) \int_0^{\infty} ds \chi(u-s) e^{-i\omega s} [W_0^A(u, u-s) - W_0^B(u, u-s)], \quad (3.17)$$

where  $W_0^A$  and  $W_0^B$  are the pull-backs of the unregularised Wightman distributions in the two situations. The representation (2.5) of the Wightman distribution in a Hadamard state guarantees that the divergences in (3.17) cancel and the integral is well defined. This is particularly convenient for numerical calculations.

## 4 Sharp switching limit

In this section we discuss the response function (3.16) in the limit of sharp switch-on and switch-off. As in the case of Minkowski vacuum [8], we shall isolate the divergence due to the sharp switching from a finite remainder and show that a well-defined notion of instantaneous transition rate can be defined in an appropriate limit.

To control the switch-on and switch-off, we assume the switching function to have the form<sup>3</sup>

$$\chi(u) = h_1 \left( \frac{u - \tau_0 + \delta}{\delta} \right) \times h_2 \left( \frac{-u + \tau + \delta}{\delta} \right), \quad (4.1)$$

where  $\tau$ ,  $\tau_0$  and  $\delta$  are parameters satisfying  $\tau_0 < \tau$  and  $0 < \delta$ , and  $h_i$ ,  $i = 1, 2$ , are non-negative  $C^\infty$  functions satisfying  $h_i(x) = 0$  for  $x \leq 0$  and  $h_i(x) = 1$  for  $1 \leq x$ . This means that the detector is turned on smoothly during the interval  $(\tau_0 - \delta, \tau_0)$ , with a profile determined by the function  $h_1$ , it then remains turned on at constant coupling

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<sup>3</sup>Our formula (4.1) corrects a typographical error in the argument of  $h_1$  in equation (4.1) of [8].

strength for the time  $\Delta\tau := \tau - \tau_0$ , and it is finally turned off smoothly during the interval  $(\tau, \tau + \delta)$ , with a profile determined by the function  $h_2$ . The functions  $h_i$  are regarded as fixed. We initially regard  $\tau_0$  and  $\tau$  as fixed but will eventually allow  $\tau$  to vary.

The first term in (3.16) is equal to  $-(\omega/4\pi)(\Delta\tau + \delta C_1)$ , where  $C_1$  is a positive constant. The second term in (3.16) was analysed in [8], with the result that it only depends on  $\Delta\tau$  and  $\delta$  through the combination  $\delta/\Delta\tau$  and has at small  $\delta/\Delta\tau$  the asymptotic form

$$\frac{1}{2\pi^2} \ln \left( \frac{\Delta\tau}{\delta} \right) + C_2 + O \left( \frac{\delta}{\Delta\tau} \right), \quad (4.2)$$

where  $C_2$  is a constant and the full expansion of the  $O$ -term proceeds in positive powers of  $\delta/\Delta\tau$ . The last term in (3.16) can be analysed by breaking the integrations into the various subintervals as in [8], with the result

$$2 \int_{\tau_0}^{\tau} du \int_0^{u-\tau_0} ds \operatorname{Re} \left[ e^{-i\omega s} W_0(u, u-s) + \frac{1}{4\pi^2 s^2} \right] + O(\delta). \quad (4.3)$$

The qualitatively new feature compared with [8] is the logarithmic singularity in  $W_0$ , but the contribution from this singularity can be verified to be of order  $O(\delta^2 \ln \delta)$  and hence subleading in (4.3). Collecting, we find

$$F(\omega) = -\frac{\omega}{4\pi} \Delta\tau + 2 \int_{\tau_0}^{\tau} du \int_0^{u-\tau_0} ds \operatorname{Re} \left[ e^{-i\omega s} W_0(u, u-s) + \frac{1}{4\pi^2 s^2} \right] + \frac{1}{2\pi^2} \ln \left( \frac{\Delta\tau}{\delta} \right) + C_2 + O(\delta). \quad (4.4)$$

By the smoothness of the spacetime and the trajectory, we may differentiate the trajectory-dependent contribution (4.3) with respect to  $\tau$  termwise, and the same holds for the trajectory-independent contributions by their explicit structure. We may therefore take the  $\tau$ -derivative in (4.4) termwise, with the result

$$\dot{F}_\tau(\omega) = -\frac{\omega}{4\pi} + 2 \int_0^{\Delta\tau} ds \operatorname{Re} \left[ e^{-i\omega s} W_0(\tau, \tau-s) + \frac{1}{4\pi^2 s^2} \right] + \frac{1}{2\pi^2 \Delta\tau} + O(\delta). \quad (4.5)$$

Equations (4.4) and (4.5) are our main result. The transition probability (4.4) diverges as  $\delta \rightarrow 0$ , but the divergence has been isolated into an explicit logarithmic term that is independent of the trajectory or the quantum state of the field. The  $\tau$ -derivative of the transition probability is given by (4.5) and remains finite as  $\delta \rightarrow 0$ . Equation (4.5) provides a definition of what is meant by the detector's transition rate, without the need to introduce spatial profiles or other regulators. In the special case of a massless field in Minkowski spacetime, in the Minkowski vacuum, the limit  $\delta \rightarrow 0$  in (4.5) recovers the transition rate obtained in [7] via a spatial profile regularisation.

We end the section with two comments. First, if we interpret the naive expression (2.10) for the transition rate as

$$\dot{F}_\tau(\omega) = \lim_{\epsilon \rightarrow 0} 2 \operatorname{Re} \int_0^{\Delta\tau} ds e^{-i\omega s} W_\epsilon(\tau, \tau - s) \quad (4.6)$$

and apply the methods of section 3, we find that (4.6) is equal to the  $\delta \rightarrow 0$  limit of our transition rate (4.5) plus an additional term proportional to  $\ddot{T}$ . The additional term vanishes if the pull-back of  $T$  to the trajectory is an affine function of  $\tau$ . The transition rate of a sharply-switched detector can thus be calculated equivalently from the  $\delta \rightarrow 0$  limit in (4.5), where the singularity in the Wightman distribution is cancelled by an explicit counterterm, or from the  $\epsilon \rightarrow 0$  limit in (4.6), provided the time function used to regularise (4.6) is an affine function of  $\tau$  on the trajectory.

Second, if we compare two detectors in different quantum states of the field, on different trajectories or even in different spacetimes, but having been in operation for the same length of proper time, we recover for the difference of the transition rates in the  $\delta \rightarrow 0$  limit the formula

$$\Delta \dot{F}_\tau(\omega) = 2 \operatorname{Re} \int_0^{\Delta\tau} ds e^{-i\omega s} [W_0^A(\tau, \tau - s) - W_0^B(\tau, \tau - s)] . \quad (4.7)$$

The difference in the transition rates in two given situations can thus be written as a Fourier transform of a function that requires no regularisation. Formula (4.7) is useful for both analytical and numerical calculations, especially in cases where the transition rate in one of the two situations is already known. In the next two sections we shall apply this formula to two such examples.

## 5 Inertial detector in the Rindler vacuum

In this section we consider a detector moving inertially through the Rindler wedge in Minkowski space, coupled to a massless scalar field in its Rindler vacuum state. This is the state in which the uniformly accelerated detectors associated with the Rindler wedge do not get excited. A naive application of the equivalence principle could be argued to imply that as an accelerated detector moving through the ‘unaccelerated’ (Minkowski) vacuum state gets excited thermally, an unaccelerated detector moving through the ‘accelerated’ (Rindler) vacuum should also get excited thermally. Working in the limit  $\delta \rightarrow 0$ , we shall show that this does not hold: the detector does have a nontrivial transition rate, but the rate is neither thermal nor constant in the detector’s proper time, and it diverges as the detector approaches the Rindler horizon.

Let  $(t, x, y, z)$  be a set of standard Minkowski coordinates in Minkowski space. We take the detector to move in the ‘right-hand-side’ Rindler wedge,  $x > |t|$ , denoted by  $R$ . The Rindler vacuum Wightman distribution  $W^R(x, x')$  in  $R$  reads [23]

$$W^R(x, x') = W^M(x, x') - \int_{-\infty}^{\infty} \frac{dv}{\pi^2 + v^2} W^M(x, x''(v)) , \quad (5.1)$$

where  $W^M(\mathbf{x}, \mathbf{y}) = [4\pi^2(\mathbf{x} - \mathbf{y})^2]^{-1}$  is the Minkowski vacuum Wightman distribution, the points  $\mathbf{x} = (t, x, y, z)$  and  $\mathbf{x}' = (t', x', y', z')$  are in  $R$  and  $\mathbf{x}''(v) := (-t' \cosh v - x' \sinh v, -x' \cosh v - t' \sinh v, y', z')$ . The difference of  $W^R(\mathbf{x}, \mathbf{x}')$  and  $W^M(\mathbf{x}, \mathbf{x}')$  consists thus of the integral of  $W^M$  over an orbit of the associated Killing vector in the *opposite* Rindler wedge. Note that since the two Rindler wedges are spacelike separated, the difference term is a nonsingular function, and the distributional character of  $W^R(\mathbf{x}, \mathbf{x}')$  comes entirely from the first term on the right-hand side in (5.1). As we shall be using formula (4.7), we are suppressing the distributional issues in (5.1).

We take the trajectory of the detector to be

$$\mathbf{x}(\tau) = (\tau, X, 0, 0), \quad (5.2)$$

where  $X$  is a positive constant and  $\tau$  is the proper time. The trajectory stays in  $R$  during the proper time interval  $|\tau| < X$ . We must therefore consider the detector response in a finite proper time interval.

We shall compute the transition rate from formula (4.7), using the Minkowski vacuum as reference state. Unlike the infinite  $\Delta\tau$  case, in which the excitation rate is zero, a detector moving inertially through Minkowski vacuum over a finite proper time  $\Delta\tau$  does have transient excitations due to the switching. This transition rate has been found in several papers (see e.g. [16]) and equals

$$\dot{F}_{\Delta\tau}^M(\omega) = -\frac{\omega}{4\pi} + \frac{\cos(\omega \Delta\tau)}{2\pi^2 \Delta\tau} + \frac{1}{2\pi^2} \omega \text{Si}(\omega \Delta\tau) \quad (5.3)$$

where Si is the sine integral function.  $\dot{F}_{\Delta\tau}^M(\omega)$  diverges as  $\Delta\tau \rightarrow 0$  (which is an artifact of omitting the  $O(\delta)$  terms in (4.5)) and approaches  $[\omega/(2\pi)]\Theta(-\omega)$  for large  $\Delta\tau$ .

Using (4.7) with (5.1), and substituting the trajectory (5.2) with  $-X < \tau_0 < \tau < X$ , the transition rate in the Rindler vacuum at time  $\tau$  can be written as

$$\dot{F}_{\tau}^R(\omega) = \dot{F}_{\Delta\tau}^M(\omega) + \Delta\dot{F}_{\tau}(\omega), \quad (5.4)$$

where

$$\begin{aligned} \Delta\dot{F}_{\tau}(\omega) = & -\frac{1}{2\pi^2} \int_0^{\Delta\tau} ds \cos(\omega s) \int_{-\infty}^{\infty} \frac{dv}{\pi^2 + v^2} \\ & \times \frac{1}{-[\tau + (\tau - s) \cosh v + X \sinh v]^2 + [X + X \cosh v + (\tau - s) \sinh v]^2}. \end{aligned} \quad (5.5)$$

The  $v$ -integral in (5.5) can be done by contour integration. Closing the contour in the upper half-plane, there are two infinite series of contributing poles, at respectively  $v = \ln(X+\tau) - \ln(X+\tau-s) + (2m+1)i\pi$  and  $v = \ln(X-\tau+s) - \ln(X-\tau) + (2m+1)i\pi$  with  $m = 0, 1, 2, \dots$ , and a single contributing pole at  $v = i\pi$ . Summing over the residues, we find

$$\Delta\dot{F}_{\tau}(\omega) = \frac{1}{2\pi^2} \int_0^{\Delta\tau} ds \frac{\cos(\omega s)}{s^2} \left[ 1 - \frac{s}{2\tau - s} \left( \frac{1}{\ln\left(\frac{X-\tau}{X-\tau+s}\right)} + \frac{1}{\ln\left(\frac{X+\tau}{X+\tau-s}\right)} \right) \right]. \quad (5.6)$$

Note that the integrand in (5.6) remains finite as  $s \rightarrow 0$ , and the integral is well defined. Regarding  $\tau_0$  fixed and  $\tau$  as variable, we see that  $\Delta\dot{F}_\tau(\omega)$  tends to zero as  $\tau \rightarrow \tau_0$ , but it diverges to  $-\infty$  as  $\tau \rightarrow X$ , which is the limit in which the trajectory approaches the Rindler horizon. We shall show in the Appendix that the asymptotic form of  $\Delta\dot{F}_\tau(\omega)$  at  $\tau \rightarrow X$  is

$$\Delta\dot{F}_\tau(\omega) = \frac{1}{2\pi^2 X} \left\{ \frac{1}{4} \ln \left( 1 - \frac{\tau}{X} \right) + \frac{1}{2} \ln \left[ -\ln \left( 1 - \frac{\tau}{X} \right) \right] + O(1) \right\}, \quad (5.7)$$

and the divergence is thus logarithmic in  $\tau$ .

The response of inertial detectors in the Rindler vacuum was previously studied by Candelas and Sciamia [24], but in a somewhat different framework. Candelas and Sciamia investigate the whole family of trajectories (5.2), and they compute the transition rate on each trajectory at the point  $\tau = \sqrt{X^2 - a^{-2}}$ , where  $a$  is a fixed positive constant. This means that the inertial trajectories are being compared along a Rindler trajectory of acceleration  $a$ . In the limit  $X \rightarrow \infty$ , it is found that the response approaches the Minkowski vacuum value  $[\omega/(2\pi)]\Theta(-\omega)$ . Candelas and Sciamia's interpretation of this result is that “for the case of a charge moving inertially in Minkowski space-time through an *accelerated* vacuum the spectrum of field fluctuations perceived by the charge *is the same as if the vacuum were unaccelerated*” ([24], p. 1717). They view the limit of large  $X$  as a means to eliminate the transient effects due to the detector starting its inertial motion at  $\tau = 0$ .

In our view these transient effects are already contained in the Minkowski vacuum part (5.3) of the response, and the additional Rindler vacuum contribution (5.6) shows that the inertial detector in the Rindler vacuum responds genuinely differently than in the Minkowski vacuum. In particular, the additional Rindler vacuum contribution (5.6) diverges as the trajectory approaches the Rindler horizon. This divergence was found previously by Davies and Ottewill [25], both by a numerical evaluation of the response and by a comparison of the response with the expectation value of  $\phi^2$ . The divergence can indeed be expected on the grounds that in the Rindler vacuum the expectation values of both  $\phi^2$  and the stress-energy tensor diverge on the horizon [23, 25, 26].

## 6 Detector at rest in Newtonian gravitational field

In this section we shall find the response of an Unruh-DeWitt detector at rest in a static, Newtonian gravitational field: a static, asymptotically flat spacetime that satisfies the linearised Einstein equations with pressureless matter as source. The quantum field is assumed massless, but with arbitrary curvature coupling, and its vacuum state is taken to be the Boulware-like vacuum defined in terms of the global timelike Killing vector. We shall see that in this situation the detector's excitation rate is always zero, but the de-excitation rate (response for negative  $\omega$ ) in general differs from that of an inertial detector in the Minkowski vacuum in Minkowski space, and the gravitational correction depends on the details of the mass distribution even when the detector is far from the

source. We also provide an order-of-magnitude estimate for this gravitational correction in atomic physics decay rates on the Earth's surface.

## 6.1 General matter distribution

The metric takes the form  $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ , where  $\eta_{\mu\nu}$  is the Minkowski metric and  $h_{\mu\nu}$  is the linearised correction. We use a system of Minkowski coordinates  $(t, \mathbf{x})$  in which  $\eta_{\mu\nu} dx^\mu dx^\nu = -dt^2 + d\mathbf{x}^2$  and the correction components  $h_{\mu\nu}$  are independent of  $t$ . We assume  $h_{\mu\nu}$  to be small enough for validity of linearised Einstein's equations, and at large  $r := |\mathbf{x}|$  we assume  $h_{\mu\nu}$  to have the asymptotically flat falloff  $O(r^{-1})$ . We further take  $h_{\mu\nu}$  to be in the Lorentz gauge,  $\partial_\mu \bar{h}^{\mu\nu} = 0$ , where  $\bar{h}^{\mu\nu} := h^{\mu\nu} - \frac{1}{2}\eta^{\mu\nu} h^\alpha{}_\alpha$ .

We need the Wightman distribution in this spacetime, in the Boulware-like vacuum that reduces to the Minkowski vacuum at the asymptotically flat infinity. Working in perturbation theory to the order that is consistent with linearised Einstein's equations, this Wightman distribution must be the sum of the Minkowski vacuum contribution  $[4\pi^2(\Delta\mathbf{x})^2]^{-1}$  and a correction  $W^{(1)}(\mathbf{x}, \mathbf{x}')$  that is first-order in  $h_{\mu\nu}$  and dies off at infinity. Restricting the attention to a detector that remains at constant  $\mathbf{x}$  and was switched on in the infinite past, equation (4.7) shows that the transition rate reads

$$\dot{F}(\omega) = -\frac{\omega}{2\pi}\Theta(-\omega) + 2\operatorname{Re}\int_0^\infty ds e^{-i\omega s} W^{(1)}(t, \mathbf{x}; t-s, \mathbf{x}). \quad (6.1)$$

The first-order Wightman distribution  $W^{(1)}$  in (6.1) depends on  $\mathbf{x}$  and  $s$  but not on  $t$ , and from (3.1c) we see that it has at  $s = 0$  an integrable logarithmic singularity proportional to the Ricci scalar.

To find  $W^{(1)}(\mathbf{x}, \mathbf{x}')$ , we first calculate the Feynman Green's function  $G_F(\mathbf{x}, \mathbf{x}')$  to first order in  $h_{\mu\nu}$ , adapting the procedure that was introduced in [27] in the context of vacuum polarisation. We then find  $W^{(1)}(\mathbf{x}, \mathbf{x}')$  from the relation

$$iG_F(\mathbf{x}, \mathbf{x}') = W(\mathbf{x}, \mathbf{x}')\Theta(t-t') + W(\mathbf{x}', \mathbf{x})\Theta(t'-t), \quad (6.2)$$

which is reliable order by order in perturbation theory as long as no new singularities turn up, and this will be seen to be the case for the first-order contributions.

Consider the equation satisfied by  $G_F(\mathbf{x}, \mathbf{x}')$ ,

$$[\Box_{\mathbf{x}} - \xi R(\mathbf{x})] G_F(\mathbf{x}, \mathbf{x}') = \frac{1}{\sqrt{-g(\mathbf{x})}} \delta(\mathbf{x}, \mathbf{x}'), \quad (6.3)$$

and expand  $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$  and  $G_F(\mathbf{x}, \mathbf{x}') = G_F^{(0)}(\mathbf{x}, \mathbf{x}') + G_F^{(1)}(\mathbf{x}, \mathbf{x}')$ , where  $G_F^{(0)}$  is the Minkowski vacuum Feynman propagator,

$$G_F^{(0)}(\mathbf{x}, \mathbf{x}') = \frac{-i}{4\pi^2 [-(t-t')^2 + |\mathbf{x} - \mathbf{x}'|^2 + i\epsilon]}, \quad (6.4)$$



with its distributional part specified by the prescription  $\epsilon \rightarrow 0_+$ . Dropping second order terms and noting that  $G_F^{(0)}$  satisfies the zeroth-order equation, we obtain

$$\begin{aligned}\square_x^{(0)} G_F^{(1)}(\mathbf{x}, \mathbf{x}') &= [\partial_\mu (\bar{h}^{\mu\nu} \partial_\nu) + \xi R^{(1)}(\mathbf{x})] G_F^{(0)}(\mathbf{x}, \mathbf{x}') \\ &= [\bar{h}^{00} \partial_t^2 + \xi R^{(1)}(\mathbf{x})] G_F^{(0)}(\mathbf{x}, \mathbf{x}'),\end{aligned}\quad (6.5)$$

where the last equality follows because in the Lorentz gauge  $\bar{h}_{\mu\nu}$  satisfies the Minkowski space wave equation

$$\square^{(0)} \bar{h}_{\mu\nu} = -16\pi G T_{\mu\nu} \quad (6.6)$$

and we are assuming static, pressureless matter.

The key observation now is that because the spacetime is static and asymptotically flat,  $G_F$  can be regarded as the analytic continuation of the *unique* Green's function on the positive definite section. As this holds order by order in perturbation theory, we can solve (6.5) for  $G_F^{(1)}$  by using  $G_F^{(0)}$  as the inverse of  $\square^{(0)}$ , with the result

$$\begin{aligned}G_F^{(1)}(\mathbf{x}, \mathbf{x}') &= \int d\tilde{\mathbf{x}} G_F^{(0)}(\mathbf{x}, \tilde{\mathbf{x}}) [\bar{h}^{00}(\tilde{\mathbf{x}}) \partial_t^2 + \xi R^{(1)}(\tilde{\mathbf{x}})] G_F^{(0)}(\tilde{\mathbf{x}}, \mathbf{x}') \\ &= -\frac{1}{16\pi^4} \int d\tilde{\mathbf{x}} \frac{1}{[-(t - \tilde{t})^2 + |\mathbf{x} - \tilde{\mathbf{x}}|^2 + i\epsilon]} \\ &\quad \times [\bar{h}^{00}(\tilde{\mathbf{x}}) \partial_t^2 + \xi R^{(1)}(\tilde{\mathbf{x}})] \frac{1}{[-(t' - \tilde{t})^2 + |\mathbf{x}' - \tilde{\mathbf{x}}|^2 + i\epsilon]}.\end{aligned}\quad (6.7)$$

The integral over  $\tilde{t}$  can be done by residues, using the time-independence of  $R^{(1)}$  and  $\bar{h}^{00}$ . Defining  $G_F^{(1)}(s, \mathbf{x}) := G_F^{(1)}(t, \mathbf{x}; t - s, \mathbf{x})$ , and writing  $X := |\mathbf{x} - \tilde{\mathbf{x}}|$  for short, we obtain

$$G_F^{(1)}(s, \mathbf{x}) = \frac{-i}{8\pi^3} \int \frac{d\tilde{\mathbf{x}}}{\sqrt{X^2 + i\epsilon}} \left[ \frac{\xi R^{(1)}(\tilde{\mathbf{x}})}{s^2 - 4(X^2 + i\epsilon)} + \frac{2\bar{h}^{00}(\tilde{\mathbf{x}}) [3s^2 + 4(X^2 + i\epsilon)]}{[s^2 - 4(X^2 + i\epsilon)]^3} \right]. \quad (6.8)$$

As  $\nabla^2 \bar{h}^{00} = -2R^{(1)}$ , where  $\nabla^2 := \partial_{x_1}^2 + \partial_{x_2}^2 + \partial_{x_3}^2$ , and as  $h_{\mu\nu}$  has the falloff  $O(r^{-1})$ , we can integrate the term involving  $\bar{h}^{00}$  by parts and take  $R^{(1)}$  as a common factor. Assuming  $s \neq 0$ , and dropping terms that go to zero as  $\epsilon \rightarrow 0$ , we obtain

$$\begin{aligned}G_F^{(1)}(s, \mathbf{x}) &= \frac{-i}{8\pi^3} \int d\tilde{\mathbf{x}} R^{(1)}(\tilde{\mathbf{x}}) \left[ \frac{2\xi - 1}{2X [(s^2 - 4i\epsilon) - 4X^2]} + \frac{1}{X(s^2 - 4i\epsilon)} \right. \\ &\quad \left. + \frac{1}{(s^2 - 4i\epsilon)^{3/2}} \ln \left( \frac{2X - \sqrt{s^2 - 4i\epsilon}}{2X + \sqrt{s^2 - 4i\epsilon}} \right) \right].\end{aligned}\quad (6.9)$$

Note that the  $O(r^{-1})$  falloff of  $h_{\mu\nu}$  implies that  $R^{(1)}$  has the falloff  $O(r^{-3})$ , and the integral in (6.9) hence converges pointwise in  $s$ . As a consistency check, we also note

that if  $R^{(1)}$  vanishes in a neighbourhood of the point  $\mathbf{x}$ , taking in (6.9) the limit  $s \rightarrow 0$  and  $\epsilon \rightarrow 0$  yields a nonsingular expression that reproduces the vacuum polarisation  $\langle \phi^2(\mathbf{x}) \rangle$  that was found for this class of spacetimes in [27].

We now assume  $s > 0$ . Let  $G_{F,1}^{(1)}(s, \mathbf{x})$  and  $G_{F,2}^{(1)}(s, \mathbf{x})$  denote the contributions to  $G_F^{(1)}(s, \mathbf{x})$  from respectively the first term and the last two terms in the integral in (6.9). Taking the limit  $\epsilon \rightarrow 0$  in  $G_{F,2}^{(1)}(s, \mathbf{x})$  is elementary, with the result

$$G_{F,2}^{(1)}(s, \mathbf{x}) = -\frac{i}{2\pi^2} \int_0^\infty dX \tilde{R}(X) \left\{ \frac{X}{s^2} + \frac{X^2}{s^3} \left[ \ln \left( \frac{|2X-s|}{2X+s} \right) + i\pi \Theta \left( \frac{s}{2X} - 1 \right) \right] \right\}, \quad (6.10)$$

where  $\tilde{R}(X)$  denotes the average of  $R^{(1)}$  over a sphere of radius  $X$  about  $\mathbf{x}$  and the dependence of  $\tilde{R}(X)$  on  $\mathbf{x}$  is suppressed. In  $G_{F,1}^{(1)}(s, \mathbf{x})$ , splitting the integrand into partial fractions and taking the limit  $\epsilon \rightarrow 0$  yields

$$G_{F,1}^{(1)}(s, \mathbf{x}) = \frac{2\xi - 1}{32\pi} \left[ \tilde{R}(s/2) + i(H\tilde{R})(s/2) + i(H\tilde{R})(-s/2) \right], \quad (6.11)$$

where  $H$  stands for the Hilbert transform,

$$(Hf)(x) := \frac{1}{\pi} P \int_{-\infty}^\infty dy \frac{f(y)}{y-x}, \quad (6.12)$$

with  $P$  denoting the principal value integral, and  $\tilde{R}$  is understood to vanish for negative argument. Recall now that the Hilbert transform can be written as  $Hf = -if_+ + if_-$ , where  $f_+$  and  $f_-$  are respectively the projections of  $f$  to the positive and negative frequency subspaces,  $f_+(x) := (2\pi)^{-1/2} \int_0^\infty e^{-i\omega x} \hat{f}(\omega) d\omega$ ,  $f_-(x) := (2\pi)^{-1/2} \int_{-\infty}^0 e^{-i\omega x} \hat{f}(\omega) d\omega$ , and  $\hat{f}$  denotes the Fourier transform,  $\hat{f}(\omega) = (2\pi)^{-1/2} \int_{-\infty}^\infty e^{i\omega x} f(x) dx$  [28]. As  $\tilde{R}$  vanishes for negative argument, it thus follows from (6.11) that

$$G_{F,1}^{(1)}(s, \mathbf{x}) = \frac{2\xi - 1}{16\pi} \left[ \tilde{R}_+(s/2) + \tilde{R}_+(-s/2) \right]. \quad (6.13)$$

Let now  $\Delta \dot{F}_{\mathbf{x}}(\omega) := \dot{F}(\omega) + [(\omega/(2\pi))\Theta(-\omega)]$  denote the correction to the Minkowski space transition rate. Using (6.1) and (6.2), we can write  $\Delta \dot{F}_{\mathbf{x}}(\omega)$  in terms of  $G_{F,1}^{(1)}$  and  $G_{F,2}^{(1)}$  as

$$\Delta \dot{F}_{\mathbf{x}}(\omega) = 2 \operatorname{Re} \int_0^\infty ds e^{-i\omega s} \left[ iG_{F,1}^{(1)}(s, \mathbf{x}) + iG_{F,2}^{(1)}(s, \mathbf{x}) \right]. \quad (6.14)$$

The contribution to  $\Delta\dot{F}_{\mathbf{x}}(\omega)$  from  $G_{F,1}^{(1)}$  equals

$$\begin{aligned}
\Delta\dot{F}_{\mathbf{x},1}(\omega) &= -\frac{2\xi-1}{8\pi} \text{Im} \int_0^\infty ds e^{-i\omega s} \left[ \tilde{R}_+(s/2) + \tilde{R}_+(-s/2) \right] \\
&= -\frac{2\xi-1}{8\pi} \text{Im} \int_{-\infty}^\infty ds e^{-i\omega s} \tilde{R}_+(s/2) \\
&= -\frac{2\xi-1}{8\pi} \Theta(-\omega) \text{Im} \int_{-\infty}^\infty ds e^{-i\omega s} \tilde{R}(s/2) \\
&= -\frac{2\xi-1}{4\pi} \Theta(-\omega) \text{Im} \int_0^\infty dX e^{-2i\omega X} \tilde{R}(X) \\
&= \frac{2\xi-1}{16\pi^2} \Theta(-\omega) \int d\tilde{\mathbf{x}} R^{(1)}(\tilde{\mathbf{x}}) \frac{\sin(2\omega X)}{X^2}, \tag{6.15}
\end{aligned}$$

where we have used the definition of the positive frequency projection, changed the integration variable to  $X = s/2$  and finally written  $\tilde{R}$  in terms of  $R^{(1)}$ . To evaluate the contribution to  $\Delta\dot{F}_{\mathbf{x}}(\omega)$  from  $G_{F,2}^{(1)}$ , we interchange the integrals over  $s$  and  $X$ , justified by the absolute convergence of the double integral, and obtain

$$\begin{aligned}
\Delta\dot{F}_{\mathbf{x},2}(\omega) &= \frac{1}{\pi^2} \int_0^\infty dX \tilde{R}(X) \\
&\quad \times \text{Re} \int_0^\infty ds e^{-i\omega s} \left\{ \frac{X}{s^2} + \frac{X^2}{s^3} \left[ \ln \left( \frac{|2X-s|}{2X+s} \right) + i\pi \Theta\left(\frac{s}{2X} - 1\right) \right] \right\}. \tag{6.16}
\end{aligned}$$

The integral over  $s$  in (6.16) may be interpreted as the integral of a complex analytic function along the positive real axis, with a contour deformation to the lower half-plane near the logarithmic singularity at  $s = 2X$ . For  $\omega > 0$ , the contour can be deformed to the negative imaginary axis, and the integral vanishes on taking the real part. For  $\omega < 0$ , the contour can be deformed to that shown in Figure 1. The contribution from the large arc vanishes when the arc is taken to infinity, and the contribution from the positive imaginary axis vanishes on taking the real part. The only nonvanishing contribution comes from the branch cut at  $s > 2X$ . Collecting, we find

$$\begin{aligned}
\Delta\dot{F}_{\mathbf{x},2}(\omega) &= \frac{2}{\pi} \Theta(-\omega) \int_0^\infty dX X^2 \tilde{R}(X) \int_{2X}^\infty ds \frac{\sin(\omega s)}{s^3} \\
&= \frac{1}{2\pi^2} \Theta(-\omega) \int d\tilde{\mathbf{x}} R^{(1)}(\tilde{\mathbf{x}}) \int_{2X}^\infty ds \frac{\sin(\omega s)}{s^3}. \tag{6.17}
\end{aligned}$$

If desired, the integral over  $s$  in (6.17) can be expressed as a sum of elementary functions and the sine integral function. The form in (6.17) is however more convenient for the observations that we shall make below.

Combining (6.15) and (6.17), and using the linearised Einstein equation (6.6) to write  $R^{(1)} = 8\pi G\rho$ , where  $\rho$  is the matter density, the gravitational correction to the

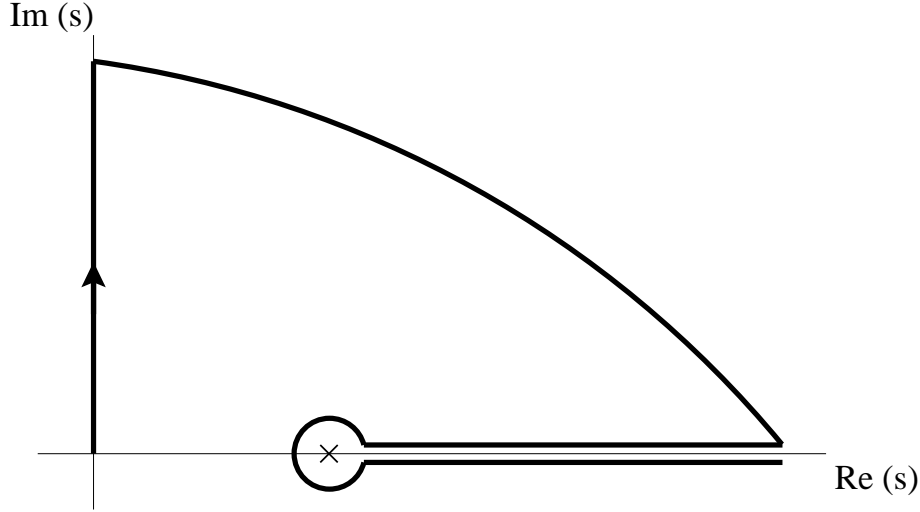


Figure 1: The contour in the complex  $s$  plane for evaluating the  $s$ -integral in (6.16) for  $\omega < 0$ . The branch point at  $s = 2X$  is indicated by a cross and the cut is at  $s > 2X$ .

Minkowski space transition rate takes the final form

$$\Delta \dot{F}_{\mathbf{x}}(\omega) = \frac{G}{2\pi} \Theta(-\omega) \int d\tilde{\mathbf{x}} \rho(\tilde{\mathbf{x}}) \left[ (2\xi - 1) \frac{\sin(2\omega X)}{X^2} + 8 \int_{2X}^{\infty} ds \frac{\sin(\omega s)}{s^3} \right], \quad (6.18)$$

where we recall that  $X$  is defined by  $X := |\mathbf{x} - \tilde{\mathbf{x}}|$ . As a check, we note that the integral in (6.18) converges in absolute value: the quantity in the brackets is of order  $O(X^{-2})$  as  $X \rightarrow \infty$  and of order  $O(X^{-1})$  as  $X \rightarrow 0$ , and  $\rho(\mathbf{x}) = O(|\mathbf{x}|^{-3})$  at  $|\mathbf{x}| \rightarrow \infty$ .

Two observations on the gravitational correction (6.18) are immediate. First, the correction vanishes for  $\omega > 0$ . The excitation rate is thus zero: *a static Newtonian gravitational field causes no excitations in a static detector*.

Second, the correction depends on the matter distribution within the source even when the source is compact and we consider the leading behaviour far from the source,  $|\mathbf{x}| \rightarrow \infty$ . Classical intuition might suggest that in this limit the de-excitation rate should only depend on the monopole moment of the mass distribution, the total mass. However, as the second term in the brackets in (6.18) is of order  $O(X^{-3})$  at  $X \rightarrow \infty$ , the leading contribution at  $|\mathbf{x}| \rightarrow \infty$  comes entirely from the first term in the brackets and can be evaluated, with the result

$$\begin{aligned} \Delta \dot{F}_{\mathbf{x}}(\omega) = \frac{G(2\xi - 1)}{2\pi|\mathbf{x}|^2} \Theta(-\omega) & \left[ \sin(2\omega|\mathbf{x}|) \int d\tilde{\mathbf{x}} \rho(\tilde{\mathbf{x}}) \cos(2\omega \hat{\mathbf{x}} \cdot \tilde{\mathbf{x}}) \right. \\ & \left. - \cos(2\omega|\mathbf{x}|) \int d\tilde{\mathbf{x}} \rho(\tilde{\mathbf{x}}) \sin(2\omega \hat{\mathbf{x}} \cdot \tilde{\mathbf{x}}) \right] + O(|\mathbf{x}|^{-3}), \end{aligned} \quad (6.19)$$

where  $\hat{\mathbf{x}} := \mathbf{x}/|\mathbf{x}|$ . If  $|\omega R_0| \ll 1$ , where  $R_0$  is the characteristic length scale of the source, the square brackets in (6.19) can be approximated by  $M \sin(2\omega|\mathbf{x}|)$ , where  $M$

is the total mass, but outside this limit the square brackets depend also on the higher multipole moments of the source. It is worth mentioning that when  $R_0$  is a typical stellar or planetary scale and  $\omega$  is a typical atomic frequency, we in fact have  $|\omega R_0| \gg 1$ , in which limit the integrals in (6.19) can be estimated by WKB techniques. The de-excitation rate of an atom far from a star therefore carries an imprint of the internal structure of the star.

## 6.2 Constant density star

As an example, we consider a spherical star of constant density  $\rho_0$  and radius  $R_0$ . The Ricci scalar has now a discontinuity at the surface of the star, and this spacetime therefore falls outside the smooth setting in which we have been working. We suspect that the non-smoothness is a technical issue that will not have a significant effect on the particle detector and shall proceed, with due caution.

The integrals in (6.18) can be evaluated in terms of the sine integral function. When  $\mathbf{x}$  is outside the star,  $r := |\mathbf{x}| > R_0$ , we find

$$\begin{aligned} \Delta \dot{F}_r(\omega) = & \frac{G\rho_0}{4\omega^2 r} \Theta(-\omega) \left[ \xi \left( 2\omega(r + R_0) \cos[2\omega(r - R_0)] - 2\omega(r - R_0) \cos[2\omega(r + R_0)] \right. \right. \\ & + \sin[2\omega(r - R_0)] - \sin[2\omega(r + R_0)] \\ & + 4\omega^2(r^2 - R_0^2) \left( \text{Si}[2\omega(r - R_0)] - \text{Si}[2\omega(r + R_0)] \right) \Big) \\ & + \frac{1}{6} \left( 64\pi\omega^4 r R_0^3 - \left( 2\omega(r + 3R_0) - 4\omega^3(r^3 + r^2 R_0 - 5r R_0^2 + 3R_0^3) \right) \cos[2\omega(r - R_0)] \right. \\ & + \left( 2\omega(r - 3R_0) + 4\omega^3(-r^3 + r^2 R_0 + 5r R_0^2 + 3R_0^3) \right) \cos[2\omega(r + R_0)] \\ & + \left( -3 + 2\omega^2(r^2 + 2r R_0 - 3R_0^2) \right) \sin[2\omega(r - R_0)] \\ & + \left( 3 + 2\omega^2(-r^2 + 2r R_0 - 3R_0^2) \right) \sin[2\omega(r + R_0)] \\ & \left. \left. + 8\omega^4(r - R_0)^3(r + 3R_0) \text{Si}[2\omega(r - R_0)] - 8\omega^4(r + R_0)^3(r - 3R_0) \text{Si}[2\omega(r + R_0)] \right) \right]. \end{aligned} \quad (6.20)$$

Consider in particular the limit in which  $\mathbf{x}$  approaches the surface of the star,  $r \rightarrow R_0$ : this presumably is the situation with the best experimental prospects of observing the gravitational correction to the de-excitation rate. We find

$$\begin{aligned} \Delta \dot{F}_{R_0}(\omega) = & \frac{G\rho_0}{48\omega^2 R_0} \Theta(-\omega) \left[ 4\omega R_0(8\omega^2 R_0^2 - 1) \cos(4\omega R_0) + (3 - 12\xi + 8\omega^2 R_0^2) \sin(4\omega R_0) \right. \\ & \left. + 8\omega R_0(8\pi\omega^3 R_0^3 + 6\xi - 1) + 128\omega^4 R_0^4 \text{Si}(2\omega R_0) \right]. \end{aligned} \quad (6.21)$$

For  $|\omega R_0| \gg 1$ , the asymptotic behaviour of (6.21) is

$$\Delta \dot{F}_{R_0}(\omega) \sim \frac{G\rho_0}{\omega} \left( \xi - \frac{1}{6} \right) \Theta(-\omega). \quad (6.22)$$

If  $\rho_0$  is the density of the Earth and  $\omega$  is a typical atomic frequency, the ratio of (6.22) to the Minkowski vacuum transition rate  $[\omega/(2\pi)]\Theta(-\omega)$  is of order  $(G\rho_0/\omega^2) \sim 10^{-42}$ . If the transition rate for the electromagnetic field behaves qualitatively similarly to that in our scalar field model, we conclude that the gravitational correction to decay rates in atomic physics laboratory experiments is unobservably small.

## 7 Conclusions

In this paper we have discussed the instantaneous transition rate of an Unruh-DeWitt detector that is coupled to a scalar field in an arbitrary Hadamard state in curved spacetime. We started with a detector that is switched on and off smoothly, in which case the detector's response function  $F(\omega)$  is well defined and can be expressed as the integral of a Wightman distribution with a standard  $i\epsilon$  regulator. We showed that the limit  $\epsilon \rightarrow 0$  can be taken explicitly, with the result

$$\begin{aligned} F(\omega) = & -\frac{\omega}{4\pi} \int_{-\infty}^{\infty} du [\chi(u)]^2 + \frac{1}{2\pi^2} \int_0^{\infty} \frac{ds}{s^2} \int_{-\infty}^{\infty} du \chi(u) [\chi(u) - \chi(u-s)] \\ & + 2 \int_{-\infty}^{\infty} du \chi(u) \int_0^{\infty} ds \chi(u-s) \operatorname{Re} \left[ e^{-i\omega s} W_0(u, u-s) + \frac{1}{4\pi^2 s^2} \right], \end{aligned} \quad (7.1)$$

where  $\chi$  is the switching function and  $W_0$  is the pull-back of the Wightman distribution to the detector's world line, with the  $\epsilon$ -regulator having been taken pointwise to zero. We then showed that when the switch-on and switch-off have a fixed shape but take each place within the time interval  $\delta$ , the sharp switching limit  $\delta \rightarrow 0$  results into a logarithmic divergence in  $F(\omega)$ , but the derivative of  $F(\omega)$  with respect to the total detection time  $\Delta\tau$  remains finite and is given by

$$\dot{F}_\tau(\omega) = -\frac{\omega}{4\pi} + 2 \int_0^{\Delta\tau} ds \operatorname{Re} \left[ e^{-i\omega s} W_0(\tau, \tau-s) + \frac{1}{4\pi^2 s^2} \right] + \frac{1}{2\pi^2 \Delta\tau} + O(\delta). \quad (7.2)$$

As a consequence, the difference  $\Delta \dot{F}_\tau(\omega)$  in the transition rates of two detectors in different quantum states, on different trajectories and even in different spacetimes, but having the same switching function, has a well defined  $\delta \rightarrow 0$  limit, given by

$$\Delta \dot{F}_\tau(\omega) = 2 \operatorname{Re} \int_0^{\Delta\tau} ds e^{-i\omega s} [W_0^A(\tau, \tau-s) - W_0^B(\tau, \tau-s)]. \quad (7.3)$$

The case of a detector switched on in the infinite past can be defined by the  $\Delta\tau \rightarrow \infty$  limit in (7.2) and (7.3), subject to suitable asymptotic conditions. We emphasise that

all the integrals in the above formulas are integrals of ordinary functions, no longer involving  $i\epsilon$  regulators or other distributional aspects. These results generalise to the setting of general Hadamard states in curved spacetime the results that were obtained for the massless field in the Minkowski vacuum in [8].

We applied the difference formula (7.3) to two situations in which the reference state can be conveniently chosen to be an inertial detector in the Minkowski vacuum. First, we considered an inertial detector coupled to a massless field in the Rindler vacuum in Minkowski space, finding that the transition rate diverges logarithmically as the detector approaches the Rindler horizon. Second, we considered a detector at rest in a static, Newtonian gravitational field, coupled to a massless field with arbitrary curvature coupling, in the Boulware-like vacuum defined with respect to the global timelike Killing vector. We found the excitation rate to be zero, but the de-excitation rate acquires a gravitational correction that depends on the details of the mass distribution within the source, even in the limit in which the source is compact and the detector is far from the source. Using a spherical constant density mass distribution as an example, we estimated the gravitational corrections to decay rates in atomic physics laboratory experiments on the surface of the Earth to be suppressed by 42 orders of magnitude.

A technical assumption throughout the paper was that both the spacetime and the detector trajectory were taken smooth. Given that the final formulas (7.1) and (7.2) remain well defined whenever the trajectory is sufficiently differentiable for the  $s^{-2}$  term to subtract the non-integrable part in  $W_0$ , it is tempting to suspect that the smoothness assumption on the trajectory could be relaxed. To investigate this question, two steps would need to be addressed. First, in section 2 we justified the use of the  $i\epsilon$ -regulator in the pull-back of the Wightman distribution in (2.6) by the theorems of [14, 15], which are formulated for smooth submanifolds: how do these theorems generalise to a lower degree of differentiability?<sup>4</sup> Second, to obtain in section 3 the estimates in (3.11), we assumed a trajectory that is  $C^8$  and has a suitably bounded remainder term in the Taylor expansion (say,  $C^9$  would suffice): could the techniques of section 3 be improved to relax this assumption?

Our results provide tools for investigating a particle detector's response in time-dependent situations in curved spacetime. One set of questions with which these tools could prove useful are thermal effects in black hole spacetimes in the time-dependent setting. For example, a detector that is falling freely into a static black hole in a Boulware-type vacuum [29] would be expected to have a divergent response at the horizon, in analogy with the Rindler vacuum analysis in our section 5, but might any thermal characteristics survive in the response of a detector falling through the horizon in an Unruh-type state [1] or in a Hartle-Hawking-Israel type state [4, 30]? From a complementary angle, consider the spacetime of a collapsing star, in a quantum state that was Boulware-type in the distant past: how does the response of a detector at a fixed position outside the star evolve from that found in section 6 to the thermal response? In particular, what is the time scale of this evolution, and how is the detector's response

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<sup>4</sup>This issue arises already in the Minkowski vacuum analysis in [8].

in this situation related to the the outoing energy flux that develops, or to any notion of ‘particles’ in the associated Bogoliubov transformation? On a more speculative note, might there be a relation between the response of a detector and dynamical or evolving horizons [31]?

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## A Appendix: Divergence of the transition rate at the Rindler horizon

In this appendix we verify the logarithmically divergent asymptotic form (5.7) for the difference  $\Delta\dot{F}_\tau(\omega)$  of the inertial detector transition rates in the Rindler vacuum and the Minkowski vacuum as the detector approaches the Rindler horizon.

We start from formula (5.6) for  $\Delta\dot{F}_\tau(\omega)$ . Writing  $\tau = X(1 - \epsilon)$  and scaling the integration variable in (5.6) by  $s \rightarrow Xs$ , we have

$$\Delta\dot{F}_\tau(\omega) = \frac{1}{2\pi^2 X} \int_0^{\alpha-\epsilon} ds \frac{\cos(\beta s)}{s^2} \left[ 1 + \frac{s}{2(1-\epsilon)-s} \left( \frac{1}{\ln(1+\frac{s}{\epsilon})} + \frac{1}{\ln(1-\frac{s}{2-\epsilon})} \right) \right], \quad (\text{A.1})$$

where  $\beta := \omega X$  and  $\alpha := 1 - (\tau_0/X)$ . We shall find the asymptotic form of (A.1) for  $\epsilon \rightarrow 0_+$  with fixed  $X$ ,  $\alpha$  and  $\beta$ . Note that  $0 < \alpha < 2$ .

We note first that we may replace the upper limit of the integral in (A.1) by  $\alpha$  at the expense of an error of order  $O(\epsilon)$ . To handle the remaining integral, we split the interval  $(0, \alpha)$  into the subintervals  $(0, \eta)$  and  $(\eta, \alpha)$ , where  $\eta := \sqrt{\epsilon}$ . We denote the contributions from the two subintervals by respectively  $(2\pi^2 X)^{-1} I_1$  and  $(2\pi^2 X)^{-1} I_2$  and provide separate estimates for each. A key tool for controlling the logarithms will be the Laurent expansion

$$\frac{1}{\ln(1+x)} = \frac{1}{x} + \frac{1}{2} + O(x). \quad (\text{A.2})$$

Consider  $I_1$ . We introduce the new integration variable  $r := s/\eta$ , with the range  $0 < r < 1$ . We replace the second logarithm in the integrand by the first two terms



in (A.2), at the expense of an error of order  $O(\eta)$  in  $I_1$ , and find

$$I_1 = \int_0^1 \frac{\cos(\beta\eta r) dr}{r[2(1-\eta^2) - \eta r]} \left( \frac{1}{\ln\left(1 + \frac{r}{\eta}\right)} - \frac{\eta}{r} - \frac{1}{2} \right) + O(\eta). \quad (\text{A.3})$$

As the quantity in the large parentheses in (A.3) is bounded, we may make the replacements  $\cos(\beta\eta r) \rightarrow 1$  and  $[2(1-\eta^2) - \eta r]^{-1} \rightarrow [2(1-\eta^2)]^{-1}$  at the expense of respective errors of order  $O(\eta^2)$  and  $O(\eta)$ . Changing the integration variable to  $y := r/\eta$ , we then obtain

$$I_1 = \frac{1}{2(1-\eta^2)} \int_0^{1/\eta} \frac{dy}{y} \left( \frac{1}{\ln(1+y)} - \frac{1}{y} - \frac{1}{2} \right) + O(\eta). \quad (\text{A.4})$$

We now concentrate on the divergent part. With errors of order  $O(1)$ , we first replace the lower limit of integration in (A.4) by 1, then drop the  $y^{-2}$  term and make the replacement  $[y \ln(1+y)]^{-1} \rightarrow [(1+y) \ln(1+y)]^{-1}$ . The remaining integral is elementary, with the result

$$I_1 = \frac{1}{8} \ln \epsilon + \frac{1}{2} \ln(-\ln \epsilon) + O(1). \quad (\text{A.5})$$

Consider then  $I_2$ . In the integrand shown in (A.1), we replace the argument of the second logarithm by  $(1 - \frac{s}{2})$ , the fraction  $s/[2(1-\epsilon) - s]$  by  $s/(2-s)$  and the argument of the first logarithm by  $s/\epsilon$ , with elementary estimates showing that each step produces in  $I_2$  an error of order  $O(\eta)$ , and obtain

$$I_2 = \int_\eta^\alpha ds \frac{\cos(\beta s)}{s^2} \left[ 1 + \frac{s}{2-s} \left( \frac{1}{\ln(s/\eta^2)} + \frac{1}{\ln(1 - \frac{s}{2})} \right) \right] + O(\eta). \quad (\text{A.6})$$

The terms in (A.6) that do not involve  $\ln(s/\eta^2)$  can be handled by a straightforward small  $s$  Laurent expansion, and their contribution to  $I_2$  is  $\frac{1}{4} \ln \eta + O(1)$ . In the term involving  $\ln(s/\eta^2)$ , the replacements  $\cos(\beta s) \rightarrow 1$  and  $s/(2-s) \rightarrow s/2$  can be verified to produce in  $I_2$  errors of order  $O(1/(\ln \eta))$ , and the remaining integral is elementary and evaluates to  $O(1)$ . Hence

$$I_2 = \frac{1}{8} \ln \epsilon + O(1). \quad (\text{A.7})$$

Combining these results, we have

$$\Delta \dot{F}_\tau(\omega) = \frac{1}{2\pi^2 X} \left[ \frac{1}{4} \ln \epsilon + \frac{1}{2} \ln(-\ln \epsilon) + O(1) \right]. \quad (\text{A.8})$$

Formula (5.7) follows by substituting  $\epsilon = 1 - (\tau/X)$ .

## References

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